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ON THE EXISTENCE OF FREE TOPOLOGICAL GROUPS

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Given a Tychonoff space X and classes $\mathbf{U}$ and $\mathbf{V}$ of topological groups, we say that a topological group $G = G(X, \mathbf{U}, \mathbf{V})$ is a free $(\mathbf{U}, \mathbf{V})$-group over X if (a) X is a subspace of G, (b) $G \in \mathbf{U}$, and (c) every continuous $f: X \to H$ with $H \in \mathbf{V}$ extends uniquely to a continuous homomorphism $\bar{f}: G \to H$. For certain classes $\mathbf{U}$ and $\mathbf{V}$, we consider the question of the existence of free $(\mathbf{U}, \mathbf{V})$-groups. Our principal results are the following. Let $\mathbf{PA}$ and $\mathbf{CA}$ denote, respectively, the class of pseudocompact Abelian groups and the class of compact Abelian groups. Then

(a) there is a free $(\mathbf{PA}, \mathbf{PA})$-group over X iff $X = \emptyset$; and

(b) there is for each X a free $(\mathbf{PA}, \mathbf{CA})$-group over X in which X is closed.

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1. Introduction

In this paper we consider only Tychonoff spaces; that is, completely regular, Hausdorff spaces. The word 'space' is to be interpreted in this way throughout. It is well known that if $\mathcal{T}$ is a Hausdorff topology for a group G such that the function $\langle x, y\rangle \to x - y$ is continuous, then $G = \langle G, \mathcal{T}\rangle$ is a space; we say in this case that G is a *topological group.*

Given spaces X and Y we set

$$\mathbf{C}(X, Y) = \{f \in Y^X : f \text{ is continuous}\},$$

and given topological groups G and H we set

$$\mathbf{H}(G, H) = \{f \in \mathbf{C}(G, H) : f \text{ is a homomorphism}\}.$$

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We denote by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the integers, rationals and reals, respectively, and by $\mathbb{T}$ the circle group. In each case the usual algebraic and topological properties are assumed. We deal almost exclusively with Abelian groups, so it is convenient to use additive notation and to denote the identity of each group by 0. (No difficulty will arise in connection with $\mathbb{T}$ if one makes the identification $\mathbb{T}=\mathbb{R}/\mathbb{Z}$.)

For G a group and $X\subseteq G$, we denote by $\langle X\rangle$ the subgroup of G generated by X. For G abelian and $X\subseteq G$, the non-0 elements of $\langle X\rangle$ have the form $\sum_{i=1}^{n} k_ix_i$ with $1\leqslant n<\omega$, with x_i distinct elements of X, and with each $k_i\in\mathbb{Z}\backslash\{0\}$. For G Abelian and $X\subseteq G$, we say that X is *independent* in G if the following condition is satisfied:

If $\sum_{i=1}^{n} k_ix_i=0$ (x_i distinct), then each $k_ix_i=0$.

When G is Abelian and $X\subseteq G$, the statement that X is independent in G and $\langle X\rangle$ is torsion-free is equivalent to the statement that the group $\langle X\rangle$ is algebraically isomorphic to the free Abelian group over X.

Modifying terminology suggested by Engelking and Mrówka [8], for a space X and $\mathbf{U}$ a class of spaces we say that X is **U**-*regular* if X is homeomorphic to a subspace of a product of elements of $\mathbf{U}$ (repetitions permitted). And when $\mathbf{U}$ is a class of topological groups we denote by $\tilde{\mathbf{U}}$ the class of all groups which are (topologically isomorphic to) a closed subgroup of a product of a set of groups in $\mathbf{U}$. We note that if $\mathbf{U}=\{\mathbb{T}\}$ then $\tilde{\mathbf{U}}$ is the class of all compact Abelian groups; this is a consequence of the fact that for every compact Abelian group G the set $\mathbf{H}(G,\mathbb{T})$ separates points of G [17] (22.17), so the usual evaluation function from G into $\mathbb{T}^{\mathbf{C}(X,\mathbb{T})}$ is an isomorphism and a homeomorphism onto its range.

For a set $\{G_i: i\in I\}$ of topological groups we write

$$\bigoplus_{i\in I} G_i=\{x\in\prod_{i\in I} G_i: |\{i\in I: x_i\neq 0\}|<\omega\},$$

and

$$\sum_{i\in I} G_i=\{x\in\prod_{i\in I} G_i: |\{i\in I: x_i\neq 0\}|\leqslant\omega\}.$$

We adopt notation as follows.

- **P**: the class of pseudocompact groups;
- **PA**: the class of pseudocompact Abelian groups;
- **C**: the class of compact groups;
- **CA**: the class of compact Abelian groups.

1.1. The reader may well be familiar with several aspects of the following theorem. The general argument, which may be viewed as an application of the adjoint functor theorem of Freyd [10], [11], has been developed and exploited in other contexts (cf. [21], [1], [27], [24], [22], [23], [17], [9], [16], [30]). For a construction of $G_{\mathbf{U}}(X)$ different from the one suggested by Kakutani [20] and Samuel [32], and for many applications, the reader may consult Thomas [35].

The earliest version of the following result (with $\mathbf{U}$ the class of all Abelian topological groups) is due to Markov [25], [26]; see also [31] and [13], [14]. Morris [29] gives a helpful, comprehensive survey of 'varieties' of topological groups and of constructions related to that of Theorem 1.3.

We note with emphasis that in the following theorem the group G constructed satisfies $G \in \tilde{\mathbf{U}}$, not necessarily $G \in \mathbf{U}$. (Thus G is not necessarily a free $(\mathbf{U}, \mathbf{U})$-group over X in the sense of the definition given in our Abstract.) This should come as no surprise: If $\mathbf{U} = \{\mathbb{R}\}$ or $\mathbf{U} = \{\mathbb{T}\}$ then every space is $\mathbf{U}$-regular, but of course not every $\mathbf{U}$-regular space X satisfies $X \subseteq G \in \mathbf{U}$.

1.2. Theorem. *Let* $\mathbf{U}$ *be a class of topological groups and let* X *be a* $\mathbf{U}$*-regular space. Then*

(1) *There is a topological group* $G = G_{\mathbf{U}}(X)$ *such that*
  (a) X *is a subspace of* G;
  (b) $G \in \tilde{\mathbf{U}}$; *and*
  (c) *every* $f \in \mathbf{C}(X, H)$ *with* $H \in \mathbf{U}$ *extends uniquely to* $\bar{f} \in \mathbf{H}(G, H)$.

(2) *For a group* G *as in* (1), *the following two conditions are equivalent*:
  (d) $\langle X \rangle$ *is dense in* G.
  (d′) *every* $f \in \mathbf{C}(X, H)$ *with* $H \in \tilde{\mathbf{U}}$ *extends uniquely to* $\bar{f} \in \mathbf{H}(G, H)$.

(3) G *may be chosen to satisfy* (d) *and* (d′).

(4) *The group* $G = G_{\mathbf{U}}(X)$ *with* (a), (b), (c) *and* (d) *is unique in the sense that if* G' *is another topological group with properties* (a), (b), (c) *and* (d) *then there is a topological isomorphism* η *from* G *onto* G' *such that* $\eta(x) = x$ *for all* $x \in X$.

Proof. (1) Let $\{\langle K_i, f_i \rangle : i \in I\}$ be a listing of all pairs with K_i a closed subgroup of an element of $\mathbf{U}$, with $f_i \in \mathbf{C}(X, K_i)$, with $\langle f_i[X] \rangle$ a dense subgroup of K_i, and with the additional technical condition that (as a set) K_i is a subset of the set $\mathcal{P}(\mathcal{P}(|X| + \omega))$. (This last condition assures us that I is a set rather than a proper class, so the product $K = \prod_{i \in I} K_i$ is a topological group.) Since X is $\mathbf{U}$-regular, the evaluation function $e: X \to K$ defined by $(ex)_i = f_i(x)$ is an homeomorphism. Identify X with $e[X]$, and let G be the closure in K of the group $\langle X \rangle$. Properties (a), (b) and (d) clearly hold.

To see (c), note that for every $f \in \mathbf{C}(X, H)$ with $H \in \mathbf{U}$ there is $i \in I$ such that $\langle \mathrm{cl}_H f[X], f \rangle$ is naturally isomorphic to the pair $\langle K_i, f_i \rangle$; thus f extends even to $g \in \mathbf{H}(K, K_i) = \mathbf{H}(K, H)$, and $\bar{f} = g | G$ is a homomorphism as required. It is clear that if $f' \in \mathbf{H}(G, H)$ and $f' | X = \bar{f} | X = f$, then $f' | \langle X \rangle = \bar{f} | \langle X \rangle$ (because f' and $\bar{f}$ are homomorphisms) and hence $f' = \bar{f}$ (because f' and $\bar{f}$ are continuous on G and $\langle X \rangle$ is dense in G); this establishes uniqueness in (c).

(2) (d)$\Rightarrow$(d′). (This is a standard argument.) Given $f \in \mathbf{C}(X, H)$ with H closed in $\prod_{i \in I} H_i$ and each $H_i \in \mathbf{U}$, set $f_i = \pi_i \circ f$, let

$$f_i \subseteq \bar{f}_i \in \mathbf{H}(G, H_i),$$

and define $\bar{f}: G \to \prod_{i \in I} H_i$ by $\bar{f}(p)_i = \bar{f}_i(p)$. Then $f \in \mathbf{H}(G, \prod_{i \in I} H_i)$, and since H is closed in $\prod_{i\in I} H_i$ and

$$\bar{f}[\langle X\rangle] = \langle \bar{f}[X]\rangle = \langle f[X]\rangle \subseteq H$$

we have $\bar{f}[G] \subseteq H$ from (d), as required.

(d')$\Rightarrow$(d). Let H be the closure of $\langle X\rangle$ in G and let i denote the inclusion function from X into H. Since $H \in \tilde{\mathbf{U}}$ there is $\bar{i} \in \mathbf{H}(G, H)$ such that $i \subseteq \bar{i}$. Now define $j: G \to G$ by

$$j(p) = p \quad (\text{all } p \in G).$$

We have $\bar{i} \in \mathbf{H}(G, G)$, $j \in \mathbf{H}(G, G)$, and

$$\bar{i}\,|\,X = j\,|\,X = i \in \mathbf{C}(X, G).$$

It follows that $\bar{i} = j$ and hence $G = H$.

(3) We have noted already that G may be chosen to satisfy (d).

(4) (This also is a familiar argument.) Let G' be another object with (a), (b), (c) and (d). The inclusion function $i: X \to X \subseteq G'$ extends to $\bar{i}: G \to G'$ and the inclusion function $h: X \to X \subseteq G$ extends to $\bar{h}: G' \to G$; the function $\eta = \bar{i} = \bar{h}^{-1}$ is then as required.

The proof is complete. □

The following consequence of Theorem 1.2, to be used in Theorem 4.3, is closely related to work of Morris [28] on co-products of free objects.

1.3. Theorem. Let $\mathbf{U}$ *be a class of Abelian topological groups, let X and Y be (disjoint)* $\mathbf{U}$*-regular spaces, and let Z be the disjoint union space $Z = X \dot{\cup} Y$. Then $G_{\mathbf{U}}(Z) = G_{\mathbf{U}}(X) \times G_{\mathbf{U}}(Y)$.*

Proof. Set $G = G_{\mathbf{U}}(X) \times G_{\mathbf{U}}(Y)$ and note $G \in \tilde{\mathbf{U}}$. Let $K(X)$ and $K(Y)$ be the groups K defined (for X and Y respectively) in the proof of Theorem 1.2(1), define $\phi: Z \to K(X) \times K(Y)$ by

$$\phi(x) = (x, 0) \text{ for } x \in X \quad \text{and} \quad \phi(y) = (0, y) \text{ for } y \in Y$$

and note that ϕ is a homeomorphism of Z onto a a subspace (also denoted Z) of G. From the relation

$$\langle Z\rangle = \langle\langle X\rangle \times \{0\} \cup \{0\} \times \langle Y\rangle\rangle = \langle X\rangle \times \langle Y\rangle$$

and the fact that $\langle X\rangle$ and $\langle Y\rangle$ are dense in $G_{\mathbf{U}}(X)$ and $G_{\mathbf{U}}(Y)$ respectively, it is immediate that $\langle Z\rangle$ is dense in G. Now let $f \in C(Z, H)$ with $H \in \mathbf{U}$, and set $f_1 = f\,|\,X$ and $f_2 = f\,|\,Y$. It is clear, denoting by $\bar{f}_1$ and $\bar{f}_2$ the continuous homeomorphisms to $G_{\mathbf{U}}(X)$ and $G_{\mathbf{U}}(Y)$ extending f_1 and f_2 respectively, and using the fact that the group H is Abelian, that the function

$$\bar{f}: \ G \to H$$

defined by $\bar{f}(p, q) = \bar{f}_1(p) \cdot \bar{f}_2(q)$ satisfies $f \subseteq \bar{f}$ and $\bar{f} \in \mathbf{H}(G, H)$; the uniqueness of $\bar{f}$ follows from the uniqueness of $\bar{f}_i$. The desired relation $G = G_{\mathrm{U}}(Z)$ then follows from Theorem 1.2(4) (applied to Z in place of X).

1.4. Remark. Theorem 1.3 does not generalize directly to disjoint union spaces $Z = \dot{\bigcup}_{i\in I} X_i$ with $|I| \geq \omega$. (Let $G_i = G_{\mathrm{U}}(X_i)$ and define $G(0)$, $G(1)$ by

$$G(0) = \bigoplus_{i\in I} G_i \subseteq \prod_{i\in I} G_i = G(1).$$

The natural embedding of Z into $G(0)$ satisfies

$$\langle Z\rangle = \bigoplus_{i\in I} \langle X_i\rangle;$$

thus $\langle Z\rangle$ is dense in $G(0)$, hence also in $G(1)$. Neither $G(0)$ nor $G(1)$ satisfies the conditions imposed on $G_{\mathrm{U}}(Z)$ in Theorem 1.2(1), however: $G(0)$ satisfies (c) but (in general) not (b), and $G(1)$ satisfies (b) but (in general) not (c).

For a concrete instance of the failure of the relation $G_{\mathrm{U}}(\dot{\bigcup}_{i\in I} X_i) = \prod_{i\in I} G_{\mathrm{U}}(X_i)$ let $I = \omega$ with $X_n = \{n\}$ for $n < \omega$, so that $\dot{\bigcup}_{n<\omega} X_n$ is the discrete space ω, take $\mathbf{U} = \{\mathbb{T}\}$, and for notational simplicity set

$$A(0) = G_{\mathrm{U}}(\omega) \quad \text{and} \quad A(1) = \prod_{n<\omega} G_{\mathrm{U}}(\{n\}).$$

We claim not only that the groups $A(i)$ are not topologically isomorphic but indeed that they are not homeomorphic as topological spaces. Let $C(i)$ be the identity component of the compact group $A(i)$, let $D(i)$ and $E(i)$ be the (discrete) Pontrjagin dual of $A(i)$ and $A(i)/C(i)$ respectively, and let $S(i)$ be the torsion subgroup of $D(i)$. It is clear (as in 3.2) that

$$D(0) = \mathbf{C}(\omega, \mathbb{T}) = \mathbb{T}^{\omega},$$

while from 23.21 of [17] it follows that

$$D(1) = \bigoplus_{n<\omega} \mathbb{T}.$$

From 24.15 and 24.20 of [17] we have

$$w(A(i)/C(i)) = |E(i)| = |S(i)|,$$

so that $w(A(0)/C(0)) = \mathfrak{c}$ and $w(A(1)/C(1)) = \omega$. It follows that $C(i)$ is a G_δ-set of $A(i)$ for $i = 1$ but not for $i = 0$, so that $A(0)$ and $A(1)$ are not homeomorphic.

2. Concerning free pseudocompact groups

The following definition is consistent with standard usage: For X a space and $\mathbf{U}$ a class of topological groups, a group G is a *free* $\mathbf{U}$-*group* over X if $X \subseteq G \in \mathbf{U}$ and every $f \in \mathbf{C}(X, H)$ with $H \in \mathbf{U}$ extends uniquely to $\bar{f} \in \mathbf{H}(G, H)$.

In this terminology, Theorem 1.2 may be summarized into the statement that for every **U**-regular space X there is a unique free $\tilde{\mathbf{U}}$-group over X in which $\langle X\rangle$ is dense.

We generalize the definition just given.

Definition. Let X be a space and let **U** and **V** be classes of topological groups. A topological group $G=G(X,\mathbf{U},\mathbf{V})$ is a *free* $(\mathbf{U},\mathbf{V})$-*group over* X if

(a) $X\subseteq G$;

(b) $G\in\mathbf{U}$; and

(c) every $f\in\mathbf{C}(X,H)$ with $H\in\mathbf{V}$ extends uniquely to $\bar{f}\in\mathbf{H}(G,H)$.

We note that it is not required in these definitions that X be closed in G.

Concerning these definitions we sound a note of caution: A free $(\mathbf{U},\mathbf{V})$-group over X is not, in general, the free group over X appropriately topologized. In the definition above we do not demand $\langle X\rangle=G$, and in fact in interesting cases the inclusion $\langle X\rangle\subseteq G$ is proper. Further, $\langle X\rangle$ itself may fail to be freely generated (in the usual algebraic sense) by X. An example: If X has a basis of open-and-closed sets then X is **U**-regular with $\mathbf{U}=\{\{0,1\}\}$, and $2p=0$ for all $p\in G_{\mathbf{U}}(X)$.

We say as usual that a topological group H is *totally bounded* if for every non-empty open subset U of H there is a finite subset S of H such that $H=\bigcup_{p\in S}(p+U)$. It is a theorem of Weil [36] that the totally bounded groups are exactly the subgroups of the compact groups. Further, when H is totally bounded there is (up to a topological isomorphism fixing H pointwise) a unique compact group K in which H is dense. We write $K=\bar{H}$ and we refer to K as the *Weil completion* [36] of H.

2.1. Lemma. *Every totally bounded topological group H is (topologically isomorphic with) a closed subgroup of a pseudocompact group G: if H is Abelian then G may be chosen Abelian.*

Proof. Let $\bar{H}$ be the Weil completion of H, let α be an uncountable cardinal number, and let Δ and $\bar{\Delta}$ denote, respectively, the diagonal of H and the diagonal of $\bar{H}$ within the product $\bar{H}^{\alpha}$. Define

$$G=\langle\Delta\cup\Sigma\bar{H}^{\alpha}\rangle\subseteq\bar{H}^{\alpha},$$

with $\Sigma\bar{H}^{\alpha}$ the Σ-product defined above. Since $\Sigma\bar{H}^{\alpha}$ is dense in $\bar{H}^{\alpha}$ and is pseudocompact (see [12], [6] and [23]), G itself is pseudocompact. It is easy to check that $\Delta=G\cap\bar{\Delta}$ so that Δ, a topological isomorph of H, is closed in G.

Any group with a dense Abelian subgroup is itself Abelian, so the second assertion in the statement follows from the first. In any case, for H Abelian the construction indicated yields $\bar{H}$ Abelian and hence G Abelian. □

Some of the essentials of the argument just given appeared in [5, (2.4)]. Recently we have learned from M.G. Tkchenko that he too has proved that every totally bounded topological group is topologically isomorphic with a closed subgroup of a pseudocompact group; we do not know if his proof is similar to the proof given above.

We turn now to the principal negative result of this paper.

2.2. Theorem. *Let* $\mathbf{U} = \mathbf{PA}$, *and let* X *be a space.*
Then there is a free $\mathbf{U}$*-group over* X *if and only if* $X = \emptyset$.

Proof. Clearly the group $\{0\}$ is a free $\mathbf{PA}$-group over X. Suppose now that $X \neq \emptyset$ and that ζX is a free $\mathbf{PA}$-group over X.

We claim first that $\langle X\rangle$ is dense in ζX. Set $N = \mathrm{cl}_{\zeta X}\langle X\rangle$, suppose $N \subsetneqq \zeta X$, let h be the canonical homomorphism from the pseudocompact group ζX onto the pseudocompact group $H = \zeta X/N$, and define $g : \zeta X \to H$ by $g(p) = 0$ (all $p \in \zeta X$). We have $g, h \in \mathbf{H}(\zeta X, H)$, $g \neq h$, and

$$g\,|\,X = h\,|\,X \in \mathbf{C}(X, H).$$

This contradiction establishes the claim.

The group ζX satisfies conditions (a), (b), (c) and (d) (on G) of Theorem 1.2. From $\langle X\rangle \subseteq \zeta X$ it follows that $\langle X\rangle$ itself is totally bounded and hence $\langle X\rangle \in \tilde{\mathbf{U}}$ from Lemma 2.1. The inclusion function $i : X \to \langle X\rangle$ extends (uniquely) to a continuous surjection $\bar{i} \in \mathbf{H}(\zeta X, \langle X\rangle)$, and since $\langle X\rangle$ is dense in ζX we then have $\langle X\rangle = \zeta X$; thus $\langle X\rangle$ is pseudocompact.

Now choose a non-torsion element ζ of $\mathbb{T}$ and define $f : X \to \mathbb{T}$ by $f(x) = \zeta$ (all $x \in X$). The extension $\bar{f} : \zeta X \to \mathbb{T}$ satisfies

$$\bar{f}[\zeta X] = \bar{f}[\langle X\rangle] = \langle\{\zeta\}\rangle.$$

The continuous image of a pseudocompact space is pseudocompact and hence $\langle\{\zeta\}\rangle$, a proper dense subgroup of $\mathbb{T}$, is psuedocompact. This contradiction completes the proof. □

3. The groups $G_{\mathbf{U}}(X)$ with $\mathbf{U} = \{\mathbb{T}\}$

The construction of 1.1 furnishes for each space X the free $\mathbf{CA}$-group over X, namely the group $G_{\{\mathbb{T}\}}(x) = G_{\mathbf{CA}}(X)$. Of course, X is closed in $G_{\mathbf{CA}}(X)$ if and only if X is compact. In this and the following section we show the existence of a free $(\mathbf{PA}, \mathbf{CA})$-group over X in which X is closed. For simplicity we write

$$FX = G_{\mathbf{CA}}(X) \text{ for each space } X.$$

We note some simple properties of the groups FX.

3.1. Lemma. *Let* X *be a space. Then*

(a) *X is closed in $\langle X\rangle$;*

(b) *X is independent in $\langle X\rangle$;*

(c) *$\langle X\rangle$ is torsion-free; and*

(d) *if D is dense in X then $\langle D\rangle$ is dense in FX.*

Proof. Let ζ be an element of $\mathbb{T}$ of infinite order.

(a) Let $p\in\langle X\rangle\backslash X$. If $p=0$ define $f(x)=\zeta$ for all $x\in X$; the extension $\bar{f}\in \mathbf{H}(FX,\mathbb{T})$ satisfies $0\notin\bar{f}^{-1}(\{\zeta\})\supseteq X$. If $p=\sum_{i=1}^n k_ix_i$ with $1\leq n<\omega$, x_i distinct elements of X, and $k_i\in\mathbb{Z}\backslash\{0\}$, set $k=\sum_{i=1}^n|k_i|$ and arrange the choice of ζ so that the 'arc' $A=[-\zeta,+\zeta]\subseteq\mathbb{T}$ does not contain $k\zeta$. Choose $f\in\mathbf{C}(X,\mathbb{T})$ so that

$$f[X]\subseteq A,$$

$$f(x_i)=\zeta\quad\text{for } k_i>0,$$

$$f(x_i)=-\zeta\quad\text{for } k_i<0.$$

Then $p\notin\bar{f}^{-1}(A)\supseteq X$.

For (b) and (c) let $p=\sum_{i=1}^n k_ix_i$ with n, x_i and k_i as above and suppose that either $p=0$ or $kp=0$ for some $k\in\mathbb{Z}\backslash\{0\}$. Choose $f\in\mathbf{C}(X,\mathbb{T})$ such that

$$f(x_1)=\zeta,\qquad f(x_i)=0\quad\text{for } 2\leq i\leq n.$$

If $p=0$ we have

$$0=\bar{f}(0)=\bar{f}\left(\sum_{i=1}^n k_ix_i\right)=\sum_{i=1}^n k_if(x_i)=k_1\zeta\neq 0;$$

the assumption $kp=0$ yields a similar contradiction.

(d) Set $K=\mathrm{cl}_{FX}\langle D\rangle$ and suppose that $K\subsetneqq FX$. According to a standard result in the theory of compact Abelian groups [1, 22.17], there is $h\in\mathbf{H}(FX,\mathbb{T})$ such that $h\equiv 0$ on K and $h\not\equiv 0$ on FX. Set $f=h|X$. From $f\equiv 0$ on D and $f\in\mathbf{C}(X,\mathbb{T})$ follows $f\equiv 0$ on X. The function $\bar{f}\in\mathbf{H}(FX,\mathbb{T})$ defined by $\bar{f}(p)=0$ (all $p\in FX$) then satisfies $\bar{f}|X=h|X$ and $\bar{f}\neq h$. This contradiction completes the proof. $\square$

3.2. Since $X\subseteq\langle X\rangle\subseteq FX\subseteq\mathbb{T}^{\mathbf{C}(X,\mathbb{T})}$ and each $f\in\mathbf{C}(X,\mathbb{T})$ extends uniquely to $\bar{f}\in\mathbf{H}(FX,\mathbb{T})$, it is natural to identify $\mathbf{C}(X,\mathbb{T})$ with $\mathbf{H}(FX,\mathbb{T})$. We give this space the discrete topology and we note that so topologized the group $\mathbf{H}(FX,\mathbb{T})$ is the Pontrjagin dual [17] of the compact group FX. For $p\in FX$ the homomorphism $\Phi(p):\mathbf{C}(X,\mathbb{T})\to\mathbb{T}$ defined by $\Phi(p)(f)=\bar{f}(p)$ is of course continuous; that is, we have $\Phi(p)\in\mathbf{H}(\mathbf{C}(X,\mathbb{T}),\mathbb{T})$. According to the Pontrjagin duality theorem [18] the function Φ maps FX onto $\mathbf{H}(\mathbf{C}(X,\mathbb{T}),\mathbb{T})$; that is, we have

$$\mathbf{H}(\mathbf{C}(X,\mathbb{T}),\mathbb{T})=\{\Phi(p): p\in FX\}.$$

3.3. It is interesting to note that the groups FX may or may not be torsion-free. For examples to this effect let us denote by $\mathbf{N}(X)$ the group of null-homotopic elements of $\mathbf{C}(X, \mathbb{T})$ and as usual by $H^1(X)$ the first cohomology group

$$H^1(X) = \mathbf{C}(X, \mathbb{T})/\mathbf{N}(X).$$

We will verify the following statements.

(i) If X is compact and $H^1(X)$ is trivial, then FX is torsion-free.

(ii) $F\mathbb{T}$ is not torsion-free.

For (i), suppose there is $p \in FX$ such that p has order $n < \omega$. Then

$$\Phi(p)[\mathbf{C}(X, \mathbb{T})] = \{k/n : 0 \leq k < n\} \subseteq [0, 1) = \mathbb{T} = \mathbb{R}/\mathbb{Z} \tag{$*$}$$

and there is $f \in \mathbf{C}(X, \mathbb{T})$ such that $\bar{f}(p) = \Phi(p)(f) = 1/n$. Since f is null-homotopic there is a $g \in \mathbf{C}(X, \mathbb{R})$ such that

$$f(x) = e^{g(x)} \mod 1 \quad \text{for all } x \in X$$

(see [33]). We set

$$g' = g/n \quad \text{and} \quad f' = e^{g'} \mod 1,$$

so that $f = nf'$. From $(*)$ it follows that

$$\Phi(p)(f) = n\Phi(p)(f') = 0,$$

a contradiction.

(ii) It is well-known that $H^1(\mathbb{T})$ is isomorphic to $\mathbb{Z}$ [34]. Consequently, since $\mathbb{Z}$ is a free group, $\mathbf{C}(\mathbb{T}, \mathbb{T})$ is isomorphic to $\mathbb{Z} \times \mathbf{N}(\mathbb{T})$. It follows from Pontrjagin duality that $F\mathbb{T}$, the dual of $\mathbf{C}(\mathbb{T}, \mathbb{T})$, is topologically isomorphic to the product of $\mathbb{T}$ with the (compact) dual of $\mathbf{N}(\mathbb{T})$ [17, 23.27(a), 23.22, 24.3].

It follows in particular from (i) that if the compact space X is contractible or zero-dimensional, then FX is torsion-free. Indeed if X is contractible then surely $H^1(X)$ is trivial, and if X is zero-dimensional then every $f \in \mathbf{C}(X, \mathbb{T})$ extends continuously over the cone over X [34], hence again lies in $\mathbf{N}(X)$.

For these and additional applications of the relation $\mathbf{C}(X, \mathbb{T}) = \mathbf{H}(FX, \mathbb{T})$ and its consequences, the reader may consult [18] and [19].

4. A pseudocompact group

Here we continue the notation introduced above and for every space X we define

$$PX = \{p \in FX : \Phi(p) \text{ has countable range}\}.$$

4.1. Lemma. *Let X be a space. Then*

(a) *PX is a subgroup of FX;*

(b) *$PX \cap \langle X \rangle = \{0\}$; and*

(c) *PX is G_δ-dense in FX.*

Proof. (a) is obvious. For (b), it is enough to show that if

$$0 \neq p = \sum_{i=1}^{n} k_i x_i \in \langle X \rangle$$

with $k_i \in \mathbb{Z}\backslash\{0\}$ and $x_i \in X$, then every point of $\mathbb{T}$ is in the range of $\Phi(p)$. To this end let $t \in \mathbb{T}$, find $s \in \mathbb{T}$ to that $k_1 s = t$, and find $f \in \mathbf{C}(X, \mathbb{T})$ so that $f(x_1) = s$ and $f(x_i) = 0$ for $2 \leqslant i \leqslant n$; then

$$\Phi(p)(f) = \bar{f}(p) = k_1 s = t,$$

as required.

(c) A (basic) G_δ-set in $\mathbb{T}^{\mathbf{C}(X,\mathbb{T})}$ has the form

$$U = \{r\} \times \mathbb{T}^{\mathbf{C}(X,\mathbb{T})\backslash C}$$

with C a countable subset of $\mathbf{C}(X, \mathbb{T})$ and $r \in \mathbb{T}^C$. In what follows we fix such U, we assume there is $q \in U \cap FX$, and we (will) show there is $p \in U \cap PX$.

Set $N = \langle C \rangle \subseteq \mathbf{C}(X, \mathbb{T})$ and let M be a subgroup of $\mathbf{C}(X, \mathbb{T})$ maximal with respect to the property that $M \cap N = \{0\}$. We claim that for every $f \in \mathbf{C}(X, \mathbb{T})$ there are $n \in \mathbb{Z}\backslash\{0\}$, $g \in M$ and $h \in N$ such that $h = nf + g$. If $f \in M$ one may take $n = 1$, $g = -f$ and $h = 0$. If $f \notin M$ there is $h \in \langle M \cup \{f\} \rangle \cap N$ with $h \neq 0$; from $h = nf + g \in N$ with $g \in M$ and $M \cap N = \{0\}$ follows $n \neq 0$, as required. The proof of the claim is complete.

Now set $A = \langle M \cup N \rangle \subseteq \mathbf{C}(X, \mathbb{T})$ and define $\Psi : A \to \mathbb{T}$ by

$$\Psi(g + h) = \Phi(q)(h) \quad \text{for } g \in M,\, h \in N.$$

Since $M \cap N = \{0\}$, the function Ψ is a well-defined homomorphism on A; a standard argument based on the divisibility of $\mathbb{T}$ shows that Ψ extends to a homomorphism (also denoted Ψ) from $\mathbf{C}(X, \mathbb{T})$ to $\mathbb{T}$. That is, we have $\Psi \in \boldsymbol{H}(\boldsymbol{C}(X, \mathbb{T}), \mathbb{T})$. As indicated earlier, there is $p \in FX$ such that $\Psi = \Phi(p)$. For $h \in C \subseteq N$ we have

$$p_h = \bar{h}(p) = \Phi(p)(h) = \Psi(h) = \Phi(q)(h) = \bar{h}(q) = q_h,$$

so that $p_C = q_C = r$ and hence $p \in U$. It remains to show $p \in PX$.

Define $E = \{t \in \mathbb{T}$: there is $n \in \mathbb{Z}\backslash\{0\}$ such that $nt \in \Phi(q)[N]\}$. Since N is countable, so is E. For every $f \in \mathbf{C}(X, \mathbb{T})$ there are $n \neq 0$, $g \in M$ and $h \in N$ as in the claim above and we have $n\Phi(p)(f) = \Phi(p)(nf) = \Phi(p)(h - g) = \Psi(h - g) = \Phi(q)(h) \in \Phi(q)[N]$ and hence $\Phi(p)(f) \in E$. Thus $p \in PX$, as required. The proof is complete. □

We observe next that the groups PX are preserved by continuous homomorphisms between groups of the form FX.

4.2. Lemma. *Let X and Y be spaces and $h \in \mathbf{H}(FX, FY)$. Then $h[PX] \subseteq PY$.*

Proof. Let $p \in PX$ and let C be the (countable) range of the function $\Phi(p)$. To show $h(p) \in PY$ it is enough to show that every $f \in \mathbf{C}(Y, \mathbb{T})$ satisfies $\Phi(h(p))(f) \in C$.

Given such f, define $g = f \circ h | X$. Then $g \in \mathbf{C}(X, \mathbb{T})$ and we have $\bar{g} \in \mathbf{H}(FX, \mathbb{T})$, $\bar{f} \circ h \in \mathbf{H}(FX, \mathbb{T})$, and

$$\bar{g} | X = \bar{f} \circ h | X = g \in \mathbf{C}(X, \mathbb{T}).$$

It follows that $\bar{g} = \bar{f} \circ h$ and we have

$$\Phi(h(p))(f) = \bar{f}(h(p)) = \bar{g}(p) = \Phi(p)(g) \in C,$$

as required. $\square$

We proceed to the principal positive result of this paper.

4.3. Theorem. *For every space X there is a topological group ξX such that*

(a) *X is embedded in ξX as a closed subspace;*

(b) *$\langle X \rangle$ is dense in ξX; and*

(c) *ξX is a free* (**PA**, **CA**)*-group over X.*

The groups ξX may be chosen so that in addition

(d) *if X and Y are spaces then every $f \in \mathbf{C}(X, Y)$ extends uniquely to $\bar{f} \in \mathbf{H}(\xi X, \xi Y)$;*

(e) *if X and Y are homeomorphic spaces then ξX and ξY are topologically isomorphic; and*

(f) *if Z is the topological disjoint union $Z = X \dot{\cup} Y$ then $\xi Z = \xi X \times \xi Y$.*

Proof. Denoting by βX the Stone–Čech compactification of X, we have $X \subseteq \langle X \rangle \subseteq F\beta X$ and $P\beta X \subseteq F\beta X$. We set

$$\xi X = \langle X \cup P\beta X \rangle \subseteq F\beta X$$

and we note that since $P\beta X \subseteq \xi X \subseteq F\beta X$ and $P\beta X$ is G_δ-dense in $F\beta X$ by Lemma 4.1(c), the group ξX is also G_δ-dense in $F\beta X$ and hence pseudocompact [4]. Statement (b) is clear, since

$$\langle X \rangle \subseteq \xi X \subseteq F\beta X$$

and $\langle X \rangle$ is dense in $F\beta X$ (by Lemma 3.1(d), with X and βX replacing D and X, respectively).

We prove (a). It follows from Lemma 3.1(b) that βX is independent in $F\beta X$ (and hence $\beta X \cap \langle X \rangle = X$). It then follows that $\beta X \cap \xi X = X$ and hence X is closed in ξX. (In detail: let

$$z \in \beta X \cap \xi X = \beta X \cap \langle X \cup P\beta X \rangle$$

and using

$$\langle X \rangle \cap P\beta X \subseteq \langle \beta X \rangle \cap P\beta X = \{0\}$$

write $z = y + p$ with $y \in \langle X \rangle$ and $p \in P\beta X$. Then $y, z \in \langle \beta X \rangle$ and hence

$$z - y = p \in \langle \beta X \rangle \cap P\beta X = \{0\},$$

so that $z = y \in \langle X \rangle \cap \beta X = X$.)

Every function $f \in \mathbf{C}(X, H)$ with $H \in \mathbf{CA}$ extends uniquely to a continuous function (again denoted f) from βX to H, hence by Theorem 1.2(1)(c) uniquely to $\bar{f} \in \mathbf{H}(F\beta X, H)$. Since βX is dense in $F\beta X$, the function $\bar{f} \mid \xi X$ is the unique continuous homomorphism from ξX to H extending f. This completes the proof of (c).

(d) Given f as in (d), there is by Theorem 1.2(1)(c) a unique $h \in \mathbf{H}(F\beta X, F\beta Y)$ such that $h \mid X = f$. We set $\bar{f} = h \mid \xi X$, and we use Lemma 4.2 to write

$$\bar{f}[\xi X] = h[\xi X] = \langle h[X] \cup h[P\beta X] \rangle \subseteq \langle Y \cup P\beta Y \rangle = \xi Y.$$

The uniqueness of h (on $F\beta X$) yields the uniqueness of $\bar{f}$ (on ξX).

(e) follows from (d).

(f) Identifying Z as before with $X \times \{0\} \cup \{0\} \times Y$ we use Theorem 1.3 to write

$$\beta Z = \beta X \times \{0\} \cup \{0\} \times \beta Y \subseteq F\beta X \times F\beta Y = F\beta Z.$$

Since $\langle Z \rangle = \langle X \rangle \times \langle Y \rangle$, to verify (f) it is enough to show $P\beta Z = P\beta X + P\beta Y$. For $(p, q) \in F\beta Z$ and $(f, g) \in \mathbf{C}(\beta X, \mathbb{T}) \times \mathbf{C}(\beta Y, \mathbb{T})$ we have

$$\Phi(p, q)(f, g) = \overline{(f, g)}(p, q) = \bar{f}(p) + \bar{g}(q) = \Phi(p)(f) + \Phi(q)(g);$$

thus (range $\Phi(p, q)$) = (range $\Phi(p)$) + (range $\Phi(q)$), and $P\beta Z = P\beta X \times P\beta Y$ is immediate. □

4.4. Remark. The foregoing theorem shows among other things that every space X embeds as a closed subspace of a pseudocompact Abelian group. This fragment of Theorem 4.3 can be established rather quickly, without recourse to the full apparatus of Theorem 4.3, as follows: Use Lemma 3.1(a) to embed X as a closed subspace of a totally bounded Abelian group $\langle X \rangle \subseteq FX$, and then appeal to Lemma 2.1.

5. Other free (PA, CA)-groups

In the preceding section we constructed, for each space X, a free (**PA**, **CA**)-group ξX over X such that X is closed in ξX and $\langle X \rangle$ is dense in ξX. It is tempting (though perhaps naive) to inquire whether ξX is the only group with these properties. Here we show that for many spaces X the answer is No. (Our construction applies to every X with $|\beta X| \leq |\mathbf{C}(X, \mathbb{T})|$—in particular, to X with $|X| = 1$. It should not be difficult to modify our reasoning so that it applies to every space X, but we are content with the present less general, more succinct, argument.) We have not counted the number of isomorphism classes which our construction yields, but each of the free (**PA**, **CA**)-groups ηX differs strikingly from ξX: Every $h \in \boldsymbol{H}(\xi X, \eta X)$ satisfies $h(p) = 0$ for all $p \in \xi X$.

5.1. Remark. Aside from the small (finite) spaces alluded to above, there are arbitrarily large spaces X which satisfy $|\beta X| \leq |\mathbf{C}(X, \mathbb{T})|$. To see this, observe first that

$$|\mathbf{C}(\beta X, \mathbb{T})| = |\mathbf{C}(X, \mathbb{T})| = |\mathbf{C}(X, \mathbb{R})| = |\mathbf{C}(\beta X, \mathbb{R})|$$

for every space X, and recall the familiar fact that every space X satisfies $|\mathbf{C}(X,\mathbb{R})| = |\mathbf{C}(X,\mathbb{R})|^{\omega}$. Now let α be an uncountable limit cardinal of countable cofinality (that is, $\alpha = \sum_{n<\omega} \alpha(n)$ with $\alpha(0)$ an arbitrary infinite cardinal and $\alpha(n+1) = 2^{\alpha(n)}$ for $n < \omega$) and let $X = \{0,1\}^{\alpha}$ or $X = \mathbb{T}^{\alpha}$. Then $|\beta X| = |X| = 2^{\alpha}$; the projection functions are continuous and hence, writing

$$|\mathbf{C}(X,\mathbb{T})| = |\mathbf{C}(X,\mathbb{R})| = \kappa \geqslant \alpha,$$

we have

$$2^{\alpha} = 2^{\sup \alpha(n)} = \prod_n 2^{\alpha(n)} \leqslant \alpha^{\omega} \leqslant \kappa^{\omega} = \kappa,$$

as asserted.

5.2. Lemma. *Let X be a space and let N be a subgroup of $\mathbf{C}(X,\mathbb{T})$ such that*

$$|N| < |\mathbf{C}(X,\mathbb{T})| = \kappa.$$

Then there is $F \subseteq \mathbf{C}(X,\mathbb{T})$ such that $|F| = \kappa$, F is independent, $\langle F\rangle$ is torsion-free, and $\langle F\rangle \cap N = \{0\}$.

Proof. Define $\Omega : \mathbf{C}(X,\mathbb{R}) \to \mathbf{C}(X,\mathbb{T})$ by

$$\Omega(f)(x) = \mathrm{e}^{f(x)},$$

and note that Ω is a homomorphism with

$$(\text{kernel } \Omega) = \{f \in \mathbf{C}(X,\mathbb{R}) : f[X] \subseteq \mathbb{Z}\}.$$

Define

$$A = \{f \in \mathbf{C}(X,\mathbb{R}) : f[X] \subseteq \mathbb{Q}\}$$

and

$$B = \{f \in \mathbf{C}(X,\mathbb{R}) : f[X] \subseteq \pi \cdot \mathbb{Q}\},$$

and note that $|A| = |B|$ and $(\text{kernel } \Omega) \subseteq A$. Since $\mathbf{C}(X,\mathbb{R})$ is a vector space over $\mathbb{Q}$ and A and B are $\mathbb{Q}$-linear subspaces with $A \cap B = \{0\}$, there is a $\mathbb{Q}$-linear subspace M of $\mathbf{C}(X,\mathbb{R})$ maximal with respect to the properties $B \subseteq M$ and $A \cap M = \{0\}$. The maximality property guarantees that $A + M = \mathbf{C}(X,\mathbb{R})$: if $f \notin A + M$ there are $q \in \mathbb{Q}$, $g \in A\backslash\{0\}$ and $m \in M$ such that $qf + m = g$, and then from $A \cap M = \{0\}$ follows $q \neq 0$ and hence

$$f = g/q - m/q \in A + M.$$

It then follows that $|M| = \kappa$ (since otherwise from $|A| \leqslant |M|$ we have

$$\kappa = |\mathbf{C}(X,\mathbb{R})| = |A + M| = |A| + |M| < \kappa).$$

Now let S be a Hamel basis for M over $\mathbb{Q}$. We have

$$|S| = \kappa, \ S \text{ is independent, and } \langle S\rangle \text{ is torsion-free.} \qquad (*)$$

It follows from $M \cap (\text{kernel } \Omega) = \{0\}$ that $\Omega \mid M$ is an isomorphism, so from $(*)$ we have

$$|\Omega[S]| = \kappa, \ \Omega[S] \text{ is independent, and } \langle \Omega[S]\rangle \text{ is torsion-free.}$$

Now write $S = \bigcup_{\xi<\kappa} S_\xi$ with each $|S_\xi| = \kappa$ and with

$$S_\xi \cap S_{\xi'} = \emptyset \text{ for } \xi < \xi' < \kappa,$$

and note from the conditions on S that

$$\langle \Omega[S_\xi]\rangle \cap \langle \Omega[S_{\xi'}]\rangle = \{0\} \text{ for } \xi < \xi' < \kappa.$$

Since $|N| < \kappa$ there is $\xi < \kappa$ such that

$$N \cap \langle \Omega[S_\xi]\rangle = \{0\};$$

the set $F = \Omega[S_\xi]$ is as required. $\square$

5.3. Lemma. *Let X be a space such that*

$$|\beta X| \leqslant |\mathbf{C}(X, \mathbb{T})| = \kappa,$$

let G be a subgroup of $F\beta X$ such that $|G| < \kappa$ and $G \cap \xi X = \{0\}$, and let U be a non-empty G_δ-subset of $F\beta X$. Then there is $p \in U$ such that

$$\langle G \cup \{p\}\rangle \cap \xi X = \{0\}.$$

Proof. There are $q \in U$ and a closed, G_δ-subgroup S of $F\beta X$ such that $q + S \subseteq U$; we write

$$S = \bigcap_{f\in N} \bar{f}^{-1}(\{0\})$$

with N a countable subgroup of $\mathbf{C}(\beta X, \mathbb{T})$ and with $\bar{f} \in \mathbf{H}(F\beta X, \mathbb{T})$ for $f \in N$. Using Lemma 5.2 we find $F \subseteq \mathbf{C}(\beta X, \mathbb{T})$ such that $|F| = \kappa$, F is independent, $\langle F\rangle$ is torsion-free, and $\langle F\rangle \cap N = \{0\}$; and using the relation $\kappa = \kappa \times \kappa \times \mathfrak{c}$ we faithfully index F in the form

$$F = \{f(\xi, \xi', \eta) : \xi < \kappa, \xi' < \kappa, \eta < \mathfrak{c}\}.$$

Using $|\langle \beta X\rangle| \leqslant \kappa$ we write $\langle \beta X\rangle = \{w_\xi : \xi < \kappa\}$, with repetitions permitted.

Now fix $\xi, \xi' < \kappa$. Choose $t(\xi, \xi', 0) \in \mathbb{T}\backslash\{0\}$ and, if $\eta < \mathfrak{c}$ and $t(\xi, \xi', \eta')$ has been defined for all $\eta' < \eta$, define

$$T(\xi, \xi', \eta) = \{\Phi(w_{\xi'}) f(\xi, \xi', \eta) - f(\xi, \xi', \eta')) \\ + nt(\xi, \xi', \eta') : \eta' < \eta, n \in \mathbb{Z}\backslash\{0\}\}$$

and choose $t(\xi, \xi', \eta) \in \mathbb{T}\backslash\{0\}$ so that every $m \in \mathbb{Z}\backslash\{0\}$ satisfies $mt(\xi, \xi', \eta) \notin T(\xi, \xi', \eta)$. (The existence of such $t(\xi, \xi', \eta)$ is immediate from the fact that

$|T(\xi, \xi', \eta)| < \mathfrak{c}$.) We note in particular that for $n \in \mathbb{Z}\backslash\{0\}$ and $\eta' < \eta$ we have

$$\begin{aligned} &nt(\xi, \xi', \eta) - \Phi(w_{\xi'})(f(\xi, \xi', \eta)) \\ &\quad \neq nt(\xi, \xi', \eta') - \Phi(w_{\xi'})(f(\xi, \xi', \eta')). \end{aligned} \tag{$*$}$$

Now for $\xi < \kappa$ define $\Psi_\xi : N \cup F \to \mathbb{T}$ by

$$\Psi_\xi(f) = \begin{cases} 0 & \text{if } f \in N, \\ 0 & \text{if } f = f(\xi'', \xi', \eta) \in F \quad \text{with } \xi'' \neq \xi, \\ t(\xi, \xi', \eta) & \text{if } f = f(\xi, \xi', \eta) \in F. \end{cases}$$

The properties of F ensure that Ψ_ξ extends to a homomorphism from $\langle N \cup F\rangle$ to $\mathbb{T}$, and since $\mathbb{T}$ is divisible this in turn extends to a homomorphism (also denoted Ψ_ξ) from $\mathbf{C}(\beta X, \mathbb{T})$ to $\mathbb{T}$. As indicated earlier, there is $p_\xi \in F\beta X$ such that $\Psi_\xi = \Phi(p_\xi)$. Of course from

$$0 = \Psi_\xi(f) = \bar{f}(p_\xi) \quad \text{for all } f \in N$$

it follows that each $p_\xi \in S$ and hence $q + p_\xi \in U$. It remains to show that there is $\xi < \kappa$ such that

$$\langle G \cup \{q + p_\xi\}\rangle \cap \boldsymbol{\xi} X = \{0\}.$$

If not, then for every $\xi < \kappa$ there are $g_\xi \in G$ and $n_\xi \in \mathbb{Z}$ such that

$$0 \neq g_\xi + n_\xi(q + p_\xi) \in \boldsymbol{\xi} X;$$

we have $n_\xi \neq 0$ since $G \cap \boldsymbol{\xi} X = \{0\}$. Since $|G \times \mathbb{Z}| < \kappa$ the function $\xi \to (g_\xi, n_\xi)$ is not one-to-one, so there are distinct $\xi, \xi'' < \kappa$, $g \in G$ and $n \in \mathbb{Z}\backslash\{0\}$ such that

$$g + n(q + p_\xi) \in \xi X \quad \text{and} \quad g + n(q + p_{\xi''}) \in \boldsymbol{\xi} X$$

and hence $np_\xi - np_{\xi''} \in \boldsymbol{\xi} X = \langle X \cup P\beta X\rangle$. Since $\langle X\rangle \cap P\beta X = \{0\}$ there are $w_{\xi'} \in \langle \beta X\rangle$ and $u \in P\beta X$ such that $np_\xi - np_{\xi''} = w_{\xi'} + u$, and from

$$u = np_\xi - np_{\xi''} - w_{\xi'}$$

we have for every $\eta < \mathfrak{c}$ (writing $f(\xi, \xi', \eta) = f$ for simplicity) that

$$\begin{aligned} \Phi(u)(f(\xi, \xi', \eta)) = \Phi(u)(f) & \\ &= n\Phi(p_\xi)(f) - n\Phi(p_{\xi''})(f) - \Phi(w_{\xi'})(f)) \\ &= nt(\xi, \xi', \eta) - n \cdot 0 - \Phi(w_{\xi'})(f) \\ &\in \text{range } \Phi(u) \end{aligned}$$

for every $\eta < \mathfrak{c}$. Since

$$\Phi(u)(f(\xi, \xi'', \eta)) \neq \Phi(u)(f(\xi, \xi'', \eta')) \quad \text{for } \eta' < \eta$$

by ($*$), we have $|\text{range } \Phi(u)| = \mathfrak{c}$ and hence $u \notin P\beta X$. This contradiction completes the proof. $\square$

5.4. Theorem. *Let X be a space such that*

$$|\beta X| \leqslant |\mathbf{C}(X, \mathbb{T})| = \kappa.$$

Then there is a dense, pseudocompact subgroup G of $F\beta X$ such that

$$|G| \leqslant \kappa \quad \textit{and} \quad G \cap \boldsymbol{\xi} X = \{0\}.$$

Proof. If $X = \emptyset$ one takes $G = \{0\}$, so we assume in what follows that $X \neq \emptyset$. From the relation

$$wF\beta X = |\mathbf{C}(\beta X, \mathbb{T})| = |\mathbf{C}(X, \mathbb{T})| = \kappa^{\omega},$$

it follows that there is a set $\{U_\xi : \xi < \kappa\}$ of non-empty G_δ-subsets of $F\beta X$ such that every non-empty G_δ-set U of $F\beta X$ contains one of the sets U_ξ.

To begin the induction, define $G(0) = \{0\}$ and use 5.3 to find $p_0 \in U_0$ such that

$$\langle G(0) \cup \{p_0\}\rangle \cap \boldsymbol{\xi} X = \{0\}.$$

Now let $0 < \xi < \kappa$ and suppose that p_η, $G(\eta)$ have been defined for all $\eta < \xi$ so that $|G(\eta)| \leqslant \kappa$, $p_\eta \in U_\eta$ and

$$\langle G(\eta) \cup \{p_\eta\}\rangle \cap \boldsymbol{\xi} X = \{0\}.$$

We define $G(\xi) = \bigcup_{\eta < \xi} G(\eta)$, we note $|G(\xi)| \leqslant \kappa$ and $G(\xi) \cap \boldsymbol{\xi} X = \{0\}$, and we use Lemma 5.3 to find $p_\xi \in U_\xi$ such that

$$\langle G(\xi) \cup \{p_\xi\}\rangle \cap \boldsymbol{\xi} X = \{0\}.$$

The group $G = \bigcup_{\xi < \kappa} G_\xi$ satisfies $|G| \leqslant \kappa$ and $G \cap \boldsymbol{\xi} X = \{0\}$; and G, since it meets every non-empty G_δ-set of $F\beta X$, is pseudocompact [5].

5.5. Theorem. *Let X be a space such that*

$$|\beta X| \leqslant |\mathbf{C}(X, \mathbb{T})| = \kappa.$$

Then there is a group $\boldsymbol{\eta} X$ such that

(a) $\boldsymbol{\eta} X \subseteq F\beta X$;

(b) *X is embedded in $\boldsymbol{\eta} X$ as a closed subspace*;

(c) *$\langle X\rangle$ is dense in $\boldsymbol{\eta} X$*;

(d) $\boldsymbol{\eta} X \cap P\beta X = \{0\}$;

(e) $|\boldsymbol{\eta} X| \leqslant \kappa$;

(f) *$\boldsymbol{\eta} X$ is a free* (**PA**, **CA**)*-group over X*; *and*

(g) $|H(\boldsymbol{\xi} X, \boldsymbol{\eta} X)| = 1$*—that is, every $h \in \mathbf{H}(\boldsymbol{\xi} X, \boldsymbol{\eta} X)$ satisfies $h(p) = 0$ for all $p \in \boldsymbol{\xi} X$.*

Proof. Choose G as in Theorem 5.4 and define

$$\boldsymbol{\eta} X = \langle X \cup G\rangle \subseteq F\beta X.$$

(a) is obvious.

(b) follows from the relation $\beta X \cap \boldsymbol{\eta} X = X$, whose proof is identical in all essential features to the proof of Theorem 4.3(a) that $\beta X \cap \boldsymbol{\xi} X = X$.

(c) follows from the relation $\langle X\rangle \subseteq \eta X \subseteq F\beta X$ and the fact that $\langle X\rangle$ is dense in $F\beta X$.

(d) is immediate from the relation

$$G \cap P\beta X = \langle X\rangle \cap P\beta X = \{0\}.$$

(e) Using Theorem 5.4 we have

$$|\eta X| \leqslant |X| + |G| \leqslant \kappa.$$

(f) The group G is dense in ηX, and $G \in \mathbf{PA}$; hence $\eta X \in \mathbf{PA}$. The proof that ηX is a free ($\mathbf{PA}$, $\mathbf{CA}$)-group over X closely parallels the corresponding proof about ξX in Theorem 4.3(c).

(g) For every $h \in \mathbf{H}(\xi X, \eta X)$ there is a unique $\bar{h} \in \mathbf{H}(F\beta X, F\beta X)$ such that $h \subseteq \bar{h}$. (This may be proved by appeal to the fact that every continuous homomorphism between totally bounded groups is uniformly continuous, or by applying Theorem 1.2 with $\mathbf{U} = \mathbf{CA}$; see also [4].) From Lemma 4.2 we have $h[P\beta X] \subseteq P\beta X$, and hence

$$h[P\beta X] \subseteq \eta X \cap P\beta X = \{0\}$$

from (d). Since h is continuous and $P\beta X$ is dense in ξX we then have $h(p) = 0$ for all $p \in \xi X$, as required. □

5.6. Remarks. (a) The preceding construction of the groups ηX was complicated by the need to arrange $G \cap P\beta X = \{0\}$ with G pseudocompact and dense in $F\beta X$. If $P\beta X$ were 'small' in $F\beta X$, this might have been achieved by a simpler argument. Let us note now that $P\beta X$ has full cardinality in $F\beta X$, that is,

$$|P\beta X| = |F\beta X| = 2^{w\beta X} = 2^{|\mathbf{C}(X,\mathbb{T})|} = 2^{\kappa}.$$

To see this apply Lemma 5.2 with $N = \{0\}$ to find a subset F of $\mathbf{C}(X, \mathbb{T})$ such that $|F| = \kappa$, F is independent, and $\langle F\rangle$ is torsion-free, and for every $E \subseteq F$ define $h_E : F \to \mathbb{T}$ by

$$h_E(f) = \begin{cases} 0 & \text{if } f \in E, \\ \frac{1}{2} & \text{if } f \in F \backslash E. \end{cases}$$

The properties of F ensure that h_E extends to a (unique) homomorphism from $\langle F\rangle$ into the subgroup $\{0, \frac{1}{2}\}$ of $\mathbb{T}$, and since the torsion subgroup $\mathbb{V}$ of $\mathbb{T}$ is divisible this in turn extends to a homomorphism (again denoted h_E) from $\mathbf{C}(X, \mathbb{T})$ to $\mathbb{V}$. As before there is $p_E \in F\beta X$ such that $h_E = \Phi(p_E)$, and since $\mathbb{V}$ is countable we have $p_E \in P\beta X$. The function $E \to p_E$ is one-to-one from $\mathscr{P}(F)$ into $P\beta X$, and we have

$$2^{\kappa} = |F\beta X| \geqslant |\xi X| \geqslant |P\beta X| \geqslant |\mathscr{P}(F)| = 2^{\kappa}.$$

(b) Theorem 5.5(g) shows among other things that for X as in Theorem 5.5 none of the groups ηX are topologically isomorphic to the group ξX. Of course, (a) above provides an alternative argument to this effect.

(c) The group ξX and the group G of Theorem 5.4 are G_δ-dense subgroups of $F\beta X$ with certain 'disjointness properties'. It is a tribute to the power of Pontrjagin duality and the isomorphism $H(F\beta X, \mathbb{T}) = \mathbf{C}(X, \mathbb{T})$ that we were able to find these groups in the manner indicated. Our initial efforts at constructing such groups, working directly in $F\beta X$, were unsuccessful. We have not attempted to recast our argument subsequently into the exclusive context of $F\beta X$, thus essentially avoiding duality theory, and we do not know whether such a project would be worthwhile or illuminating.

6. Some unsolved questions

Here we gather together a few questions, closely related to the topic of this paper, which our methods seem inadequate to solve.

6.1. As Remark 1.4 indicates, even for $\mathbf{U} = \{\mathbb{T}\}$ we have not found the correct generalization of Theorem 1.2 to spaces of the form $\dot{\bigcup}_{i\in I} X_i$ with I infinite. Theorem 4.3(g) suggests a similar challenge concerning the groups $\xi(\dot{\bigcup}_{i\in I} X_i)$. Specifically, we ask:

Let $\{X_i : i \in I\}$ be a set of (disjoint) spaces and set $Z = \dot{\bigcup}_{i\in I} X_i$.

(a) Determine the group $F\beta Z$ in terms of the groups $F\beta X_i$; and

(b) determine the group ξZ in terms of the groups ξX_i.

An example as in Remark 1.4 shows that the three relations

$$\xi Z = \bigoplus_{i\in I} \xi X_i, \qquad \xi Z = \sum_{i\in I} \xi X_i, \qquad \xi Z = \prod_{i\in I} \xi X_i$$

can all fail; we note in this connection that both $\sum_{i\in I} \xi X_i$ and $\prod_{i\in I} \xi X_i$ are pseudocompact Abelian groups [4].

6.2. It is natural to consider questions of the kind treated here for other classes of topological groups. For the sake of simplicity, let **CCA** denote the class of countably compact Abelian groups. From 1.3, every space X admits a free (**CA**, **CA**)-group; this is of course a free (**CCA**, **CA**)-group and a free (**PA**, **CA**)-group.

(a) Are there spaces with a free (**CCA**, **CCA**)-group?

(b) Does every countably compact space X admit a free (**CCA**, **CCA**)-group in which X is closed?

A positive solution to (b) would show, in particular, that every countably compact space X embeds as a closed subspace into an (Abelian) countably compact group. Since there are countably compact spaces X and Y such that $X \times Y$ is not countably compact (even with $X = Y$), this would furnish groups $G, H \in \mathbf{CCA}$ (even with $G = H$) such that $G \times H \notin \mathbf{CCA}$. Since the existence of such groups is known only in ZFC+MA [7], it may be difficult to establish a positive answer to (b). Our own efforts to answer (a) and (b) have been unsuccessful even in ZFC+MA.

6.3. The construction of Theorem 4.3 associates with every space X a group $\xi X \in \mathbf{PA}$, containing X as a closed subspace, with the property that every $f \in \mathbf{C}(X, Y)$ extends uniquely to $\bar{f} \in \mathbf{H}(\xi X, \xi Y)$. (For present purposes, let us agree to write $\bar{f} = \xi(f)$.)

The uniqueness condition assures us that if X, Y and Z are spaces and $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, Z)$, then $\xi(g \circ f) = \xi(g) \circ \xi(f)$. It is clear also, denoting by id_X the identity function on X, that $\xi(\mathrm{id}_X) = \mathrm{id}_{\xi X}$. The situation may be summarized in categorical language: ξ is a functor from the category **TYCH** of Tychonoff spaces (and continuous maps) into the category **PA** of Hausdorff pseudocompact Abelian groups (and continuous homomorphisms). Of course, ξ is not the only functor from **TYCH** to **PA**: the functor

$$X \to FX, f \to \bar{f}$$

is another (but X is not closed in FX), and the functor

$$X \to \xi X \times \mathbb{T}, f \to \xi(f) \times \mathrm{id}_{\mathbb{T}}$$

is another (but $\langle X \rangle$ is not dense in $\xi X \times \mathbb{T}$). Accordingly, we ask this question: Is there another functor $\zeta : \mathbf{TYCH} \to \mathbf{PA}$ with X closed in ζX and with $\langle X \rangle$ dense in ζX for each $X \in \mathbf{TYCH}$?

6.4. We have been concerned throughout this paper with several questions of this form: Given X, is there G such that every $f \in \mathbf{C}(X, H)$ with $H \in \mathbf{V}$ extends uniquely to $\bar{f} \in \mathbf{H}(G, H)$? If the word 'unique' is omitted, there arise a host of related questions—many of which, of course, are already answered by our results. Of those unsolved, the following is perhaps the most interesting specific problem.

(a) Does every space X embed into a group $G \in \mathbf{PA}$ in such a way that every $f \in \mathbf{C}(X, H)$ with $H \in \mathbf{PA}$ extends to $\bar{f} \in \mathbf{H}(G, H)$?

The difficulty is that there are 'too many' groups H to consider: In contrast with the device used in the proof of Theorem 1.2(1), we here see no way to replace the proper class of pseudocompact groups into which X can map continuously with a set which is adequate to our purposes. The following specific question arises.

(b) Let **CN** be the class of cardinal numbers. Is there a function $\phi : \mathbf{CN} \to \mathbf{CN}$ such that if $X \subseteq G$ with $|X| = \alpha$ and G is a pseudocompact Abelian group, then there is a pseudocompact subgroup H of G such that $X \subseteq H$ and $|H| \leq \phi(\alpha)$?

For a related question without algebraic overtones one may replace "Abelian group" and "subgroup" by "space" and "subspace", respectively. An attractive formulation runs as follows.

(c) Are there, for every two cardinals α and β, spaces X and Y such that $X \subseteq Y$, Y is pseudocompact, $|X| < \alpha$, and every pseudocompact space Y' such that $X \subseteq Y' \subseteq Y$ satisfies $|Y'| > \beta$?

[Note added August, 1987. The answer to 6.4(c) is "Yes". See [2] and [3] for a proof and generalizations.]

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